Series of random variables: Kolmogorov's three-series theorem

The goal of this problem is to prove Kolmogorov's three-series theorem. This famous theorem states necessary and sufficient conditions for the convergence of series of independent random variables:

Theorem 1 (Kolmogorov's three-series theorem). *Let $(X_n)_{n\geq 0}$ be a sequence of independent random variables.*
The series $\sum_{n\geq 0} X_n$ converges almost surely if and only if the three series

$$\sum_{n\geq 0} \mathbb{P}(|X_n| > 1), \quad \sum_{n\geq 0} \mathbb{E}[X_n 1_{|X_n|\leq 1}], \quad \textit{and} \quad \sum_{n\geq 0} \mathbb{V}(X_n 1_{|X_n|\leq 1})$$

converge.

The case of centered random variables in L^2

Let $(X_n)_{n\geq 0}$ be a sequence of independent L^2 random variables. We assume in this part that the random variables are centered (i.e. $\forall n \geq 0, \mathbb{E}[X_n] = 0$) and that the series $\sum_{n\geq 0} \mathbb{E}[X_n^2]$ converges.

For $n \geq 0$, we denote by S_n the sum $\sum_{k=0}^{n} X_k$.

Let $\epsilon > 0$. For $m \in \mathbb{N}$, we introduce $\tau_{m,\epsilon} = \inf\{k > m | |S_k - S_m| > \epsilon\}$ with the classical convention that $\inf \emptyset = +\infty$.

1. Prove that $\forall n > m, \mathbb{P}(|S_n - S_m| \geq \epsilon) \leq \frac{1}{\epsilon^2} \sum_{k=m+1}^{+\infty} \mathbb{E}[X_k^2]$.
2. For $k > m$, prove that $\mathbb{P}(\tau_{m,\epsilon} = k) \leq \frac{1}{\epsilon^2}\mathbb{E}[(S_k - S_m)^2 1_{\tau_{m,\epsilon}=k}]$.
3. Deduce that, for $n \geq k > m$, $\mathbb{P}(\tau_{m,\epsilon} = k) \leq \frac{1}{\epsilon^2}\mathbb{E}[(S_n - S_m)^2 1_{\tau_{m,\epsilon}=k}]$.
4. Deduce that $\forall m \in \mathbb{N}$,
$$\mathbb{P}(\exists k > m, |S_k - S_m| \geq \epsilon) \leq \frac{1}{\epsilon^2} \sum_{j=m+1}^{+\infty} \mathbb{E}[X_j^2].$$
5. Conclude that $(S_n)_{n\geq 0}$ converges almost surely.

A sufficient condition for convergence

Let us now consider a sequence $(X_n)_{n\geq 0}$ of independent random variables such that the three series

$$\sum_{n\geq 0} \mathbb{P}(|X_n| > 1), \quad \sum_{n\geq 0} \mathbb{E}[X_n 1_{|X_n|\leq 1}], \quad \text{and} \quad \sum_{n\geq 0} \mathbb{V}(X_n 1_{|X_n|\leq 1})$$

converge.

6. Prove that $\sum_{n\geq 0} X_n 1_{|X_n|\leq 1}$ converges almost surely.
7. Prove that almost surely $\{n \geq 0, X_n > 1\}$ is finite.
   Hint: Use Borel-Cantelli's lemma.
8. Conclude that $\sum_{n\geq 0} X_n$ converges almost surely.

Cantelli's inequality and another inequality

Let us consider a random variable $Y \in L^2$ such that $\mathbb{E}[Y] = 0$. Let us consider $\lambda > 0$.

9. Prove that
$$\mathbb{P}(Y > \lambda) \leq \inf_{y>0} \frac{\mathbb{E}[(Y + y)^2]}{(\lambda + y)^2}.$$

10. Deduce that $\mathbb{P}(Y > \lambda) \leq \frac{\mathbb{V}(Y)}{\mathbb{V}(Y)+\lambda^2}$.

Let us now consider a random variable $X \in L^2$ such that $\mathbb{E}[X] \geq 0$. Let us consider $\theta \in (0,1)$.

11. Prove that
$$\mathbb{P}(X > \theta\mathbb{E}[X]) \geq \frac{(1-\theta)^2\mathbb{E}[X]^2}{\mathbb{V}(X) + (1-\theta)^2\mathbb{E}[X]^2}.$$

Convergence of the series of variances as a necessary condition for bounded centered random variables

In this part we consider a sequence $(X_n)_{n\geq 0}$ of independent random variables. We assume they are centered (i.e. $\forall n \geq 0, \mathbb{E}[X_n] = 0$) bounded by 1 (i.e. $\forall n \geq 0, |X_n| \leq 1$) and that the series $\sum_{n\geq 0} X_n$ converges almost surely.

For $n \geq 0$, we denote by S_n the sum $\sum_{k=0}^n X_k$ and by σ_n^2 the sum $\sum_{k=0}^n \mathbb{E}[X_k^2]$.

12. Prove that
$$\forall n \geq 0, \mathbb{E}[S_n^4] \leq \sigma_n^2 + 3\sigma_n^4.$$
Hint: you can get inspiration from the classical proof of the law of large numbers for L^4 variables.

13. Deduce that
$$\forall \theta \in (0,1), \forall n \geq 0, \mathbb{P}(S_n^2 \geq \theta\sigma_n^2) \geq \frac{(1-\theta)^2\sigma_n^4}{(2+(1-\theta)^2)\sigma_n^4 + \sigma_n^2}.$$

14. Prove that if $\sum_{n=0}^{+\infty} \mathbb{E}[X_n^2] = +\infty$ then $\mathbb{P}(\sup_{n\geq 0}|S_n| = +\infty) > 0$.

15. Deduce that the series $\sum_{n\geq 0} \mathbb{E}[X_n^2]$ is convergent.

Getting rid of the centered hypothesis

In this part we consider a sequence $(X_n)_{n\geq 0}$ of independent random variables. We assume they are bounded by 1 (i.e. $\forall n \geq 0, |X_n| \leq 1$) and that the series $\sum_{n\geq 0} X_n$ converges almost surely.

16. Prove that the series $\sum_{n\geq 0} \mathbb{V}(X_n)$ is convergent.
Hint: consider $\left(\frac{X_n - X'_n}{2}\right)_{n\geq 0}$ where the two sequences $(X_n)_{n\geq 0}$ and $(X'_n)_{n\geq 0}$ are independent and identically distributed.

A necessary condition for convergence

In this part we consider a sequence $(X_n)_{n\geq 0}$ of independent random variables. We assume that the series $\sum_{n\geq 0} X_n$ converges almost surely.

17. Prove that the series $\sum_{n\geq 0} \mathbb{P}(|X_n| > 1)$ is convergent.
18. Prove that $\sum_{n\geq 0} X_n 1_{|X_n|\leq 1}$ converges almost surely.
19. Deduce that the series $\sum_{n\geq 0} \mathbb{V}(X_n 1_{|X_n|\leq 1})$ converges.
20. Deduce that the series $\sum_{n\geq 0} \mathbb{E}[X_n 1_{|X_n|\leq 1}]$ converges.
21. Conclude.