A short promenade into the world of random walks: around Polya's theorem

The goal of this problem is to compute, using elementary tools, the probability that a random walk on $\mathbb{Z}^d$ returns to its starting point. The approach developed in this problem relies heavily on power series. Toward the end, the reader may recognize the use of the celebrated Laplace's method for obtaining asymptotic equivalents of integrals with exponential integrands.

Notations

For $d \in \mathbb{N}^*$, we denote by $(e_1, \dots, e_d)$ the canonical basis of $\mathbb{R}^d$.

For each $d \in \mathbb{N}^*$, we consider a sequence $(X_{n,d})_{n\in\mathbb{N}^*}$ of i.i.d. random variables with uniform distribution in $\{e_1, -e_1, \dots, e_d, -e_d\}$. We assume that these random variables are independent.

For each $d \in \mathbb{N}^*$, we then define a random walk $(S_{n,d})_{n\in\mathbb{N}}$ by

$$\begin{cases} S_{0,d} = 0 \in \mathbb{Z}^d \\ S_{n+1,d} = S_{n,d} + X_{n+1,d}, \quad \forall n \in \mathbb{N}. \end{cases}$$

Let us introduce the following notations:

$$p_{n,d} = \mathbb{P}(S_{n,d} = 0), \quad \forall n \in \mathbb{N}, \forall d \in \mathbb{N}^*,$$

$$q_{n,d} = \mathbb{P}(S_{n,d} = 0, \forall m \in \{1, \dots, n-1\} S_{m,d} \neq 0), \quad \forall n \in \mathbb{N}^*, \forall d \in \mathbb{N}^*,$$

$$\pi_d = \mathbb{P}(\exists n \in \mathbb{N}^*, S_{n,d} = 0), \quad \forall d \in \mathbb{N}^*.$$

From $(p_{n,d})_{n\in\mathbb{N}}$ to π_d

Let us consider $d \in \mathbb{N}^*$.

Let us define the functions

$$g_d : x \in [0,1) \mapsto \sum_{n=0}^{+\infty} p_{n,d} x^n \quad \text{and} \quad h_d : x \in [0,1] \mapsto \sum_{n=1}^{+\infty} q_{n,d} x^n.$$

1. Show that g_d and h_d are well defined and continuous functions.
2. Show that if the series $\sum_{n\geq 0} p_{n,d}$ is convergent then $\lim_{x\to 1^-} g_d(x) = \sum_{n=0}^{+\infty} p_{n,d}$. Show that otherwise $\lim_{x\to 1^-} g_d(x) = +\infty$.
3. Show that $\pi_d = h_d(1)$.
4. Show that for all $n \in \mathbb{N}^*$,
$$p_{n,d} = \sum_{k=1}^{n} q_{k,d} p_{n-k,d}$$
5. Deduce that $\forall x \in [0,1), g_d(x) = 1 + g_d(x) h_d(x)$.
6. Prove that if the series $\sum_{n\geq 0} p_{n,d}$ is convergent then $\pi_d = 1 - \left(\sum_{n=0}^{+\infty} p_{n,d}\right)^{-1}$. Show that otherwise $\pi_d = 1$.

The case of dimension 1

We now consider the case $d = 1$.

7. Show that

$$p_{n,1} = \begin{cases} 0, & \text{for } n \text{ odd}, \\ \frac{C_n^{n/2}}{2^n}, & \text{for } n \text{ even}. \end{cases}$$

8. Prove that $p_{2n,1} \sim_{n\to+\infty} \frac{1}{\sqrt{n\pi}}$.
9. What is the value of π_1?

Generating functions

Let us introduce for each $d \in \mathbb{N}^*$ the function

$$G_d : x \in \mathbb{R}_+ \mapsto \sum_{n=0} p_{n,d}\frac{x^n}{n!},$$

10. Show that G_d is well defined for all $d \in \mathbb{N}^*$.
11. Prove that $\forall n \in \mathbb{N}, \forall d \in \mathbb{N} \setminus \{0, 1\}$,

$$p_{n,d} = \sum_{k=0}^{n} C_n^k \frac{1}{d^k}\left(1 - \frac{1}{d}\right)^{n-k} p_{k,1}p_{n-k,d-1}.$$

12. Deduce that $\forall d \in \mathbb{N}^*, \forall x \in \mathbb{R}_+, G_d(x) = G_1\left(\frac{x}{d}\right)^d$.
13. Prove that

$$\forall d \in \mathbb{N}^*, \forall x \in [0, 1), \quad g_d(x) = \int_0^{+\infty} G_d(tx)e^{-t}dt.$$

14. Deduce that if $t \mapsto G_1\left(\frac{t}{d}\right)^d e^{-t}$ is integrable on $\mathbb{R}_+$ then

$$\pi_d = 1 - \left(\int_0^{+\infty} G_1\left(\frac{t}{d}\right)^d e^{-t}dt\right)^{-1}$$

and that otherwise $\pi_d = 1$.

Study of G_1

15. Prove that

$$\forall x \in \mathbb{R}_+, G_1(x) = \sum_{n=0}^{+\infty} \frac{1}{n!^2}\left(\frac{x}{2}\right)^{2n}$$

16. Prove by induction that

$$\forall n \in \mathbb{N}, \int_0^{\frac{\pi}{2}} \cos^{2n}(\theta)d\theta = \frac{(2n)!}{(2^n n!)^2}\frac{\pi}{2}.$$

17. Prove that

$$\forall x \in \mathbb{R}_+, G_1(x) = \frac{1}{\pi}\int_0^{\pi} \exp(x\cos(\theta))d\theta.$$

Hint: Prove first that $\forall x \in \mathbb{R}_+, G_1(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cosh(x\cos(\theta))d\theta$.

18. Prove that

$$\forall \epsilon \in (0, 1), \exists \eta > 0, \forall \theta \in [0, \eta], 1 - \frac{\theta^2}{2}(1 + \epsilon) \leq \cos(\theta) \leq 1 - \frac{\theta^2}{2}(1 - \epsilon).$$

19. Prove that $\forall \epsilon \in (0, 1), \exists \eta > 0$,

$$\frac{e^x}{\sqrt{x}}\frac{1}{\pi\sqrt{1+\epsilon}}\int_0^{\eta\sqrt{x(1+\epsilon)}} e^{-\frac{u^2}{2}}du \leq \frac{1}{\pi}\int_0^{\pi} \exp(x\cos(\theta))d\theta \leq \frac{e^x}{\sqrt{x}}\frac{1}{\pi\sqrt{1-\epsilon}}\int_0^{\eta\sqrt{x(1-\epsilon)}} e^{-\frac{u^2}{2}}du + e^{x\cos(\eta)}.$$

20. Deduce that $G_1(x) \sim_{x \to +\infty} \frac{1}{\sqrt{2\pi}} \frac{e^x}{\sqrt{x}}$.

Polya's theorem and beyond

21. Prove that $\pi_1 = \pi_2 = 1$ and that $0 < \pi_d < 1$ for $d \geq 3$.
22. (Bonus) Prove that $\pi_d \sim_{d \to +\infty} \frac{1}{2d}$.