A short introduction to the moment problem

The goal of this problem is to touch upon the moment problem. Using the characteristic function we obtain a growth condition on moments to guarantee that the moments of a random variable characterize uniquely its distribution. We also provide a classical example showing that two random variables with different distributions can have the same moments.

The results we obtain in this problem are far from optimal. It is noteworthy that they can easily be improved using basic knowledge of complex analysis. However, deriving conditions like the famous Carleman's conditions require far more work.

The characteristic function

For any real-valued random variable X, we denote by ϕ_X the characteristic function of X defined by:

$$\forall t \in \mathbb{R}, \phi_X(t) = \mathbb{E}[\exp(itX)].$$

1. Prove that ϕ_X is a well-defined complex-valued function bounded by 1.

2. Let $p \in \mathbb{N}^*$. Prove that if $X \in L^p$ then ϕ_X is a function of class C^p with

$$\phi_X(t) = \sum_{k=0}^{p} \frac{i^k}{k!} \mathbb{E}[X^k] t^k + o(t^p).$$

The Gaussian case

Let ζ be a Gaussian $\mathcal{N}(0,1)$ random variable.

3. Prove that ϕ_ζ is a function of class C^1 and that $\forall t \in \mathbb{R}, \phi'_\zeta(t) = -t\phi_\zeta(t)$.

4. Deduce that $\forall t \in \mathbb{R}, \phi_\zeta(t) = \exp\left(-\frac{t^2}{2}\right)$.

Let ζ be a Gaussian $\mathcal{N}(\mu, \sigma^2)$ random variable.

5. Prove that $\forall t \in \mathbb{R}, \phi_\zeta(t) = \exp\left(it\mu - \frac{1}{2}\sigma^2 t^2\right)$.

Injectivity

Let Y be a real-valued random variable distributed according to a probability measure μ. Let ζ be a $\mathcal{N}(0,1)$ random variable independent of Y.

For $n \in \mathbb{N}^*$, we define $Z_n = Y + \frac{\zeta}{n}$.

Let us consider a continuous and bounded function $\psi : \mathbb{R} \to \mathbb{R}$.

6. Prove that

$$\mathbb{E}[\psi(Z_n)] = \frac{n}{\sqrt{2\pi}} \int_{z\in\mathbb{R}} \psi(z) \left(\int_{y\in\mathbb{R}} \exp\left(-\frac{1}{2}n^2(z-y)^2\right) \mu(dy) \right) dz.$$

7. Prove that
$$\forall z \in \mathbb{R}, \int_{y\in\mathbb{R}} \exp\left(-\frac{1}{2}n^2(z-y)^2\right)\mu(dy) = \int_{x\in\mathbb{R}} \exp\left(-izx\right)\frac{1}{n\sqrt{2\pi}}\exp\left(-\frac{1}{2}\frac{(z-y)^2}{n^2}\right)\phi_Y(x)dx.$$
Hint: Recall that $\exp\left(-\frac{1}{2}n^2(z-y)^2\right) = \phi_{n\zeta}(z-y) = \phi_{n\zeta}(y-z)$.

8. Prove that $\lim_{n\to+\infty}\mathbb{E}[\psi(Z_n)] = \mathbb{E}[\psi(Y)]$.

9. Deduce that
$$\int_{z\in\mathbb{R}} \psi(z)\mu(dz) = \lim_{n\to+\infty}\frac{1}{2\pi}\int_{z\in\mathbb{R}}\psi(z)\left(\int_{x\in\mathbb{R}}\exp\left(-izx\right)\exp\left(-\frac{1}{2}\frac{(z-y)^2}{n^2}\right)\phi_Y(x)dx\right)dz.$$

Let X and Y be two real-valued random variables with the same characteristic function.

10. Prove that X and Y have the same distribution.

The moment problem

Let us consider two real-valued random variables X and Y in $\bigcap_{p\in\mathbb{N}^*} L^p$ and such that
$$\forall n \in \mathbb{N}^*, \mathbb{E}[X^n] = \mathbb{E}[Y^n] =: m_n.$$

Let us assume that $\lim_{n\to+\infty}\frac{m_{2n}^{\frac{1}{2n}}}{n} = 0$.

11. Prove that
$$\lim_{n\to+\infty}\frac{\mathbb{E}[|X|^{2n+1}]^{\frac{1}{2n+1}}}{n} = 0.$$

12. Prove that the series
$$\sum_{n\geq 0}\mathbb{E}[|X|^n]\frac{|t|^n}{n!} \quad \text{and} \quad \sum_{n\geq 0}\mathbb{E}[|Y|^n]\frac{|t|^n}{n!}$$
converge for all $t\in\mathbb{R}$.

13. Deduce that X and Y have the same characteristic function.

14. Conclude.

In fact, using basic knowledge of complex analysis, the condition
$$\lim_{n\to+\infty}\frac{m_{2n}^{\frac{1}{2n}}}{n} = 0$$
can be relaxed into
$$\limsup_{n\to+\infty}\frac{m_{2n}^{\frac{1}{2n}}}{n} < +\infty.$$
A more general condition that guarantees that the moments uniquely characterize the distribution is Carleman's condition:
$$\sum_{n=1}^{+\infty} m_{2n}^{-\frac{1}{2n}} = +\infty.$$

A counterexample: the lognormal distribution

Let us consider ζ a Gaussian $\mathcal{N}(0,1)$ random variable and let us denote by X the random variable $\exp(\zeta)$.

15. Prove that for all $n \in \mathbb{N}^*, \mathbb{E}[X^n] = \exp\left(\frac{1}{2}n^2\right)$.

16. Prove that X has density $x \mapsto 1_{x>0} \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{1}{2}(\log(x))^2\right)$.

17. Prove that
$$\forall n \in \mathbb{N}, \int_0^{+\infty} x^n \frac{\sin(2\pi \log(x))}{x\sqrt{2\pi}} \exp\left(-\frac{1}{2}(\log(x))^2\right) dx = 0.$$

18. Deduce that $x \mapsto 1_{x>0} \frac{(1+\sin(2\pi \log(x)))}{x\sqrt{2\pi}} \exp\left(-\frac{1}{2}(\log(x))^2\right)$ is a probability density function and that any random variable Y with this probability density function verifies
$$\forall n \in \mathbb{N}^*, \mathbb{E}[Y^n] = \mathbb{E}[X^n].$$

19. Comment.