Dynkin's π-λ theorem and Kolmogorov's zero-one law

The second Borel-Cantelli lemma states that the probability of the limit superior of a countable family of independent events is either 0 or 1. In probability theory, several results of this kind are known as *zero–one laws*.

In this problem, we derive one of the most famous zero–one laws: **Kolmogorov's zero-one law**. Roughly speaking, it states that for a sequence of independent random variables, any *tail event* – that is, an event independent of any finite subset of the sequence – has probability either 0 or 1.

To prove Kolmogorov's zero-one law, we briefly review the basics of σ-algebras and prove the classical **Dynkin's π-λ theorem**, also sometimes known in France as the *"lemme de la classe monotone"*.

A few applications of Kolmogorov's zero-one law are presented at the end of the problem.

λ-systems, π-systems, σ-algebras and Dynkin's π-λ theorem

We consider a set Ω.

We recall that $\mathcal{B} \subset \mathcal{P}(\Omega)$ is a σ-algebra (on Ω) if and only if

- $\Omega \in \mathcal{B}$,
- $\forall B \in \mathcal{B}, B^c = \Omega \setminus B \in \mathcal{B}$,
- $\forall (B_n)_{n\in\mathbb{N}} \in \mathcal{B}^{\mathbb{N}}, \cup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

We say that $\mathcal{B} \subset \mathcal{P}(\Omega)$ is a π-system (on Ω) if and only if

- $\mathcal{B} \neq \emptyset$,
- $\forall B_1, B_2 \in \mathcal{B}, B_1 \cap B_2 \in \mathcal{B}$.

We say that $\mathcal{B} \subset \mathcal{P}(\Omega)$ is a λ-system[1] (on Ω) if and only if

- $\Omega \in \mathcal{B}$,
- $\forall B_1, B_2 \in \mathcal{B},\ B_1 \subset B_2 \implies B_2 \setminus B_1 \in \mathcal{B}$,
- $\forall (B_n)_{n\in\mathbb{N}} \in \mathcal{B}^{\mathbb{N}},\ \forall n \in \mathbb{N}, B_n \subset B_{n+1} \implies \cup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

1. Show that a σ-algebra is both a π-system and a λ-system.
2. Show that if $\mathcal{B} \subset \mathcal{P}(\Omega)$ is both a π-system and a λ-system, then it is a σ-algebra.
3. Let $\mathcal{C} \subset \mathcal{P}(\Omega)$. Show that the smallest λ-system and the smallest σ-algebra containing $\mathcal{C}$ are nonempty and respectively given by[2]

$$\lambda(\mathcal{C}) = \bigcap_{\mathcal{B}:\lambda\text{-system},\mathcal{B}\supset\mathcal{C}} \mathcal{B} \quad \text{and} \quad \sigma(\mathcal{C}) = \bigcap_{\mathcal{B}:\sigma\text{-algebra},\mathcal{B}\supset\mathcal{C}} \mathcal{B}.$$

Let $\mathcal{C} \subset \mathcal{P}(\Omega)$ be a π-system.

4. Let $C \in \mathcal{C}$. Prove that $\{A \in \mathcal{P}(\Omega), A \cap C \in \lambda(\mathcal{C})\}$ is a λ-system.
5. Deduce that $\forall B \in \lambda(\mathcal{C}), \forall C \in \mathcal{C}, B \cap C \in \lambda(\mathcal{C})$.

[1] λ-systems are also called Dynkin systems.
[2] "Smallest" here is for the order associated with the inclusion of sets.

6. Let $B \in \lambda(\mathcal{C})$. Prove that $\{A \in \mathcal{P}(\Omega), A \cap B \in \lambda(\mathcal{C})\}$ is a λ-system.
7. Deduce that $\lambda(\mathcal{C})$ is a π-system and that $\lambda(\mathcal{C}) = \sigma(\mathcal{C})$.
8. Let $\mathcal{C} \subset \mathcal{D} \subset \mathcal{P}(\Omega)$ with $\mathcal{C}$ a π-system and $\mathcal{D}$ a λ-system. Prove that $\sigma(\mathcal{C}) \subset \mathcal{D}$.[3]

Kolmogorov's zero–one law

We consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

We recall that, for a random variable X, $\sigma(X)$ denotes the smallest σ-algebra such that X is $\sigma(X)$-measurable.

We also recall that two σ-algebras $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{A}$ are independent if and only if

$$\forall B_1 \in \mathcal{B}_1, \forall B_2 \in \mathcal{B}_2, \mathbb{P}(B_1 \cap B_2) = \mathbb{P}(B_1)\mathbb{P}(B_2).$$

Let $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{A}$ be two σ-algebras. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two π-systems such that $\mathcal{B}_1 = \sigma(\mathcal{C}_1)$ and $\mathcal{B}_2 = \sigma(\mathcal{C}_2)$.

Assume that

$$\forall C_1 \in \mathcal{C}_1, \forall C_2 \in \mathcal{C}_2, \mathbb{P}(C_1 \cap C_2) = \mathbb{P}(C_1)\mathbb{P}(C_2).$$

9. Let $C_2 \in \mathcal{C}_2$. Prove that $\{B_1 \in \mathcal{B}_1, \mathbb{P}(B_1 \cap C_2) = \mathbb{P}(B_1)\mathbb{P}(C_2)\}$ is a λ-system.
10. Deduce that $\forall B_1 \in \mathcal{B}_1, \forall C_2 \in \mathcal{C}_2, \mathbb{P}(B_1 \cap C_2) = \mathbb{P}(B_1)\mathbb{P}(C_2)$.
11. Let $B_1 \in \mathcal{B}_1$. Prove that $\{B_2 \in \mathcal{B}_2, \mathbb{P}(B_1 \cap B_2) = \mathbb{P}(B_1)\mathbb{P}(B_2)\}$ is a λ-system.
12. Deduce that $\mathcal{B}_1$ and $\mathcal{B}_2$ are independent.

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$.

Let us define:

- for $n \leq m \in \mathbb{N}$, $\mathcal{B}_{n,m} = \sigma(X_n, \ldots, X_m)$,
- for $n \in \mathbb{N}$, $\mathcal{B}_{\leq n} = \mathcal{B}_{0,n}$,
- for $n \in \mathbb{N}$, $\mathcal{C}_{>n} = \bigcup_{m>n} \mathcal{B}_{n+1,m}$,
- for $n \in \mathbb{N}$, $\mathcal{B}_{>n} = \sigma(\mathcal{C}_{>n})$,
- $\mathcal{B}_\infty = \bigcap_{n\in\mathbb{N}} \mathcal{B}_{>n}$.

13. Prove that $\forall n \in \mathbb{N}$, $\mathcal{C}_{>n}$ is a π-system.
14. Prove that $\mathcal{B}_{\leq n}$ and $\mathcal{B}_{>n}$ are independent for all $n \in \mathbb{N}$.
15. Deduce that $\forall n \in \mathbb{N}$, $\mathcal{B}_{\leq n}$ and $\mathcal{B}_\infty$ are independent.
16. Deduce that $\mathcal{B}_{>0}$ and $\mathcal{B}_\infty$ are independent.
17. Conclude that $\mathcal{B}_\infty$ is independent of itself.
18. Deduce Kolmogorov's zero-one law:

$$\forall B \in \mathcal{B}_\infty, \mathbb{P}(B) \in \{0, 1\}.$$

[3]This result is called Dynkin's π-λ theorem.

Classical applications of Kolmogorov's zero-one law

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$.

19. Show that $\mathbb{P}(\{\omega \in \Omega, \lim_{n\to+\infty} X_n(\omega) \text{ exists}\})$ is either 0 or 1.
20. If $(X_n)_{n\in\mathbb{N}}$ converges almost surely, show that the limit is a constant.
21. Show that $\mathbb{P}(\{\omega \in \Omega, \sum_{n\in\mathbb{N}} X_n(\omega) \text{ converges}\})$ is either 0 or 1.