Combinatorics: number of onto mappings, Stirling numbers and Bell numbers

The goal of this problem is to derive several classical combinatorial formulas. We present two methods for determining the number of onto mappings from one finite set to another: one based on the inclusion–exclusion principle (also known as the sieve method), and another using an inversion formula attributed to Pascal.

We also explore two approaches for deriving Dobiński's formula, which gives the number of partitions of a finite set (i.e., the Bell numbers): one approach highlights the connection with the Poisson distribution, and the other uses a generating function.

Preliminaries about onto mappings

For $n, p \in \mathbb{N}^*$, let $s(n,p)$ denote the number of onto mappings from a set of n elements to a set of p elements. By convention, we also write $s(n,0) = 0$ for $n \geq 1$.

1. Compute $s(n,p)$ for $p > n$.
2. Compute $s(n,1)$ and $s(n,n)$.
3. Show that $s(n,2) = 2^n - 2$.
4. Show that $\forall n \geq 2, s(n, n-1) = \frac{1}{2}(n-1)n!$.

Inclusion–exclusion principle

In this section and the next, we denote by A the set of functions from $\{1, \ldots, n\}$ to $\{1, \ldots, p\}$.

5. What is the cardinal of A?
6. Let $A_1, \ldots, A_p$ be p sets. Prove by induction the sieve / Poncaré's formula:

$$\left| \bigcup_{1 \leq i \leq p} A_i \right| = \sum_{k=1}^{p} \sum_{1 \leq i_1 < \ldots < i_k \leq p} (-1)^{k-1} \left| \bigcap_{1 \leq j \leq k} A_{i_j} \right|.$$

7. For $i \in \{1, \ldots, p\}$, let $A_i = \{f \in A, f^{-1}(\{i\}) = \emptyset\}$. Prove that

$$s(n,p) = \left| A \setminus \bigcup_{1 \leq i \leq p} A_i \right|.$$

8. Conclude that

$$s(n,p) = \sum_{k=1}^{p} C_p^k (-1)^{p-k} k^n.$$

Inversion formula

9. Let $(a_p)_{0 \leq p \leq n}$ and $(b_p)_{0 \leq p \leq n}$ be two tuples. Show that[1]

$$\forall p \in \{0, \ldots, n\} \quad a_p = \sum_{k=0}^{p} C_p^k b_k \quad \iff \quad \forall p \in \{0, \ldots, n\} \quad b_p = \sum_{k=0}^{p} C_p^k (-1)^{p-k} a_k.$$

10. By using a partition of A, show that

$$\forall n, p \in \mathbb{N}^*, p^n = \sum_{k=1}^{p} C_p^k s(n,k).$$

[1] This inversion formula is due to Pascal.

11. Conclude again that

$$\forall n, p \in \mathbb{N}^*, s(n,p) = \sum_{k=1}^{p} C_p^k (-1)^{p-k} k^n.$$

A first route towards Dobiński formula

For $n, p \in \mathbb{N}^*$, let $S(n,p)$ denote the number of partitions of a set of n elements into p (nonempty) subsets (Stirling number of the second kind). Let also B_n denote the number of partitions of a set of n elements (Bell number).

12. Explain why $S(n,p) = \frac{s(n,p)}{p!}$.
13. Explain why $B_n = \sum_{k=1}^{\infty} S(n,k)$.
14. Show that

$$\forall n, p \in \mathbb{N}^*, p^n = \sum_{k=1}^{\infty} (p)_k S(n,k),$$

where $(p)_k = p(p-1)\dots(p-k+1)$.

15. If X is a Poisson-distributed random variable with parameter 1, show that $\mathbb{E}[(X)_k] = 1, \forall k \in \mathbb{N}^*$.
16. Deduce Dobiński formula:

$$B_n = \frac{1}{e} \sum_{k=1}^{\infty} \frac{k^n}{k!}.$$

A second route towards Dobiński formula

We extend the definition of B_n to $n = 0$ by $B_0 = 1$.

17. Show by a simple argument that $\forall n \in \mathbb{N}^*, B_n \leq n^n$.
18. Let us define

$$G : x \in [0, e^{-1}) \mapsto \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

By using Stirling formula, prove that G is well-defined and C^∞.

19. Show that $\forall n \in \mathbb{N}, B_{n+1} = \sum_{k=0}^{n} C_n^k B_k$.
20. Deduce that $G'(x) = e^x G(x)$ and that $G(x) = e^{e^x - 1}$.
21. Use a Taylor expansion of G to obtain Dobiński formula:

$$B_n = \frac{1}{e} \sum_{k=1}^{\infty} \frac{k^n}{k!}.$$

Key findings:

- The sieve / Poncaré's formula:

$$\left| \bigcup_{1 \leq i \leq p} A_i \right| = \sum_{k=1}^{p} \sum_{1 \leq i_1 < \ldots < i_k \leq p} (-1)^{k-1} \left| \bigcap_{1 \leq j \leq k} A_{i_j} \right|.$$

- The number of onto mappings from a set of cardinal n to a set of cardinal p is $s(n,p) = \sum_{k=1}^{p} C_p^k (-1)^{p-k} k^n$.

- The number B_n of partitions of a set of cardinal n is

$$B_n = \frac{1}{e}\sum_{k=1}^{\infty}\frac{k^n}{k!}.$$

- It is sometimes useful to encode a sequence into a function through power series. The resulting function, called generating function, can sometimes be computed and reveals many properties of the sequence.