Convergence in distribution: from portmanteau to Slutsky and beyond

The goal of this problem is to develop a deeper understanding of convergence in distribution, a concept that plays a central role in both probability theory and statistics. The problem explores the Portmanteau theorem and its consequences for characterizing convergence in distribution, the relationship between convergence in probability and convergence in distribution, and **Slutsky's theorem**, which is fundamental for establishing asymptotic results in statistics. As a bonus, the problem also includes **Scheffé's lemma** concerning convergence of densities.

The connection between convergence in distribution and characteristic functions is deferred to a separate problem.

Applications of the results presented here are ubiquitous in probability and statistics (see other problem sets).

Definitions

We consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ defined on this probability space. All random variables in this problem are assumed to be $\mathbb{R}^d$-valued unless otherwise stated. The norm chosen in $\mathbb{R}^d$ is denoted by $|\cdot|$.

We say that $(X_n)_{n\in\mathbb{N}}$ **converges in probability** towards a random variable X ($X_n \xrightarrow[n\to+\infty]{\mathbb{P}} X$) if and only if

$$\forall \epsilon > 0, \lim_{n\to+\infty} \mathbb{P}(|X_n - X| > \epsilon) = 0.$$

We say that $(X_n)_{n\in\mathbb{N}}$ **converges in distribution** towards a random variable[1] X ($X_n \xrightarrow[n\to+\infty]{\mathcal{D}} X$) if and only if for all bounded continuous functions ϕ, $\lim_{n\to+\infty} \mathbb{E}\left[\phi(X_n)\right] = \mathbb{E}\left[\phi(X)\right]$.

We also recall the classical notion of almost sure convergence: $(X_n)_{n\in\mathbb{N}}$ **converges almost surely** towards a random variable X ($X_n \xrightarrow[n\to+\infty]{\text{a.s.}} X$) if and only if

$$\mathbb{P}\left(\left\{\omega\in\Omega \,\middle|\, \lim_{n\to+\infty} X_n(\omega) = X(\omega)\right\}\right) = 1.$$

Portmanteau theorem

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables and X a random variable, all with values in $\mathbb{R}^d$.

The goal in this section is to prove that the following assertions are equivalent:

a) For all bounded continuous functions ϕ from $\mathbb{R}^d$ to $\mathbb{R}$,

$$\lim_{n\to+\infty} \mathbb{E}\left[\phi(X_n)\right] = \mathbb{E}\left[\phi(X)\right].$$

b) For all bounded Lipschitz functions ϕ from $\mathbb{R}^d$ to $\mathbb{R}$,

$$\lim_{n\to+\infty} \mathbb{E}\left[\phi(X_n)\right] = \mathbb{E}\left[\phi(X)\right].$$

[1] The convergence depends in fact only on the distribution of X and not on X itself.

c) For all open sets O of $\mathbb{R}^d$,

$$\liminf_{n\to+\infty} \mathbb{P}(X_n \in O) \geq \mathbb{P}(X \in O).$$

d) For all closed sets F of $\mathbb{R}^d$,

$$\limsup_{n\to+\infty} \mathbb{P}(X_n \in F) \leq \mathbb{P}(X \in F).$$

e) For all Borel sets B of $\mathbb{R}^d$ such that the frontier $\partial B = \bar{B} \setminus \mathring{B}$ verifies $\mathbb{P}(X \in \partial B) = 0$,

$$\lim_{n\to+\infty} \mathbb{P}(X_n \in B) = \mathbb{P}(X \in B).$$

1. Prove that a) implies b).

Let O be an open set of $\mathbb{R}^d$. Let us consider for all $k \in \mathbb{N}$ the function

$$\phi_k : x \in \mathbb{R}^d \mapsto \min(kd(x, O^c), 1) \quad \text{where} \quad d(x, O^c) = \inf_{y \notin O} |x - y|.$$

2. Prove that for all $k \in \mathbb{N}$ the function ϕ_k is Lipschitz.

3. Prove that for all $x \in \mathbb{R}^d$, the sequence $(\phi_k(x))_{k\in\mathbb{N}}$ is nondecreasing and converges towards $1_O(x)$.

4. Deduce that b) implies c)

5. Prove that c) and d) are equivalent.

6. Prove that c) and d) imply e).

Let Z be a random variable. We call atom of Z any point a such that $\mathbb{P}(Z = a) > 0$.

7. For any $n \in \mathbb{N}^*$, prove that there are at most n points a such that $\mathbb{P}(Z = a) \geq \frac{1}{n}$.

8. Deduce that the set of atoms of Z is at most countable.

Let ϕ be a bounded nonnegative continuous function. For all $x > 0$, let us define $B_x = \phi^{-1}((x, +\infty))$.

9. Prove that

$$\forall n \in \mathbb{N}, \quad \mathbb{E}[\phi(X_n)] = \int_0^{\|\phi\|_\infty} \mathbb{P}(X_n \in B_x)dx \quad \text{and} \quad \mathbb{E}[\phi(X)] = \int_0^{\|\phi\|_\infty} \mathbb{P}(X \in B_x)dx.$$

10. Prove that $\forall x > 0, \partial B_x \subset \phi^{-1}(\{x\})$.

11. Deduce that $\{x > 0 | \mathbb{P}(X \in \partial B_x) > 0\}$ is at most countable.

12. Conclude that e) implies that $\lim_{n\to+\infty} \mathbb{E}[\phi(X_n)] = \mathbb{E}[\phi(X)]$.

Let ϕ be a bounded continuous function.

13. Prove that e) implies that $\lim_{n\to+\infty} \mathbb{E}[\phi(X_n)] = \mathbb{E}[\phi(X)]$.

14. Conclude.

Convergence in probability vs. convergence in distribution

15. Show that if $(X_n)_{n\in\mathbb{N}}$ converges in probability towards X, then it converges in distribution towards X. *Hint: proceed by contradiction and remember that convergence in probability implies almost sure convergence up to a subsequence.*

16. Show that if $(X_n)_{n\in\mathbb{N}}$ converges in distribution towards a constant random variable X (i.e. there exists $a \in \mathbb{R}^d$ such that $\mathbb{P}(X = a) = 1$) then it converges in probability towards X.
Hint: use assertion c) or d) of portmanteau theorem.

Slutsky theorem

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}^d$-valued random variables and $(Y_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}^k$-valued random variables.

We assume that $X_n \xrightarrow[n\to+\infty]{\mathbb{P}} X$ and $Y_n \xrightarrow[n\to+\infty]{\mathcal{D}} Y$ where X and Y are two random variables with values in $\mathbb{R}^d$ and $\mathbb{R}^k$ respectively.

We also assume that X is constant (i.e. there exists $a \in \mathbb{R}^d$ such that $\mathbb{P}(X = a) = 1$).

Let ϕ be a bounded Lipschitz function from $\mathbb{R}^d \times \mathbb{R}^k$ to $\mathbb{R}$. We denote by K its Lipschitz constant.

17. Prove that $\lim_{n\to+\infty} \mathbb{E}\left[\phi(a, Y_n)\right] = \mathbb{E}\left[\phi(a, Y)\right]$.

18. Let $\epsilon > 0$. Show that $\forall n \in \mathbb{N}, |\phi(X_n, Y_n) - \phi(a, Y_n)| \leq 2\|\phi\|_\infty 1_{|X_n - a| > \epsilon} + K\epsilon$. Deduce that $\limsup_{n\to+\infty} |\mathbb{E}\left[\phi(X_n, Y_n)\right] - \mathbb{E}\left[\phi(a, Y_n)\right]| \leq K\epsilon$.

19. Conclude that $(X_n, Y_n) \xrightarrow[n\to+\infty]{\mathcal{D}} (X, Y)$.

Characterization with cumulative distribution functions

For any real-valued random variable X, we define F_X the cumulative distribution function of X by

$$F_X : x \in \mathbb{R} \mapsto \mathbb{P}(X \leq x).$$

20. Prove that F_X is nondecreasing and càdlàg.

21. Prove that the set of points where F_X is discontinuous is at most countable.

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real-valued random variables and X a real-valued random variable.

22. Prove that if $X_n \xrightarrow[n\to+\infty]{\mathcal{D}} X$, then $\lim_{n\to+\infty} F_{X_n}(x) = F_X(x)$ for all $x \in \mathbb{R}$ where F_X is continuous.

We assume now that $\lim_{n\to+\infty} F_{X_n}(x) = F_X(x)$ for all $x \in \mathbb{R}$ where F_X is continuous.

23. Prove that for all $x \in \mathbb{R}$,
$$\limsup_{n\to+\infty} \mathbb{P}(X_n \leq x) \leq \mathbb{P}(X \leq x) \quad \text{and} \quad \liminf_{n\to+\infty} \mathbb{P}(X_n < x) \geq \mathbb{P}(X < x).$$

24. Deduce that $\forall a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$ with $a < b$, $\liminf_{n\to+\infty} \mathbb{P}(X_n \in (a, b)) \geq \mathbb{P}(X \in (a, b))$.

25. Conclude that $X_n \xrightarrow[n\to+\infty]{\mathcal{D}} X$.
Hint: Use assertion c) of portmanteau theorem.

26. Conclude.

(Bonus) Characterization with density functions: Scheffé's lemma

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables and X a random variable, all with values in $\mathbb{R}^d$.

Let us consider a Borel measure μ on $\mathbb{R}^d$.

We assume that for all $n \in \mathbb{N}$, X_n has a probability density function $f_n \in L^1(\mu)$ with respect to the measure μ. We also assume that X has a probability density function $f \in L^1(\mu)$ with respect to the measure μ.

We assume that $(f_n)_{n\in\mathbb{N}}$ converges pointwise towards f.

27. Prove that $\lim_{n\to+\infty} \int_{\mathbb{R}^d} \min(f_n(x), f(x))d\mu(x) = \int_{\mathbb{R}^d} f(x)d\mu(x)$.

28. Show that $\forall a, b \in \mathbb{R}, |a-b| = a+b-2\min(a,b)$.

29. Deduce Scheffé's lemma: $(f_n)_{n\in\mathbb{N}}$ converges in $L^1(\mu)$ towards f, i.e. $\lim_{n\to+\infty} \int_{\mathbb{R}^d} |f_n(x)-f(x)|d\mu(x)$.

30. Conclude that $X_n \xrightarrow[n\to+\infty]{\mathcal{D}} X$.

Remark: this last result is mainly used for two types of μ: discrete measures and the Lebesgue measure on $\mathbb{R}^d$.