# The characteristic function: regularity and moments

The goal of this problem is to understand the link between the regularity of the characteristic function of a random variable $X$ and the moments of $X$. In particular we show the equivalence between $C^{2k}$ regularity and the existence of moments up to order $2k$. We also provide a counterexample of a characteristic function of class $C^1$ associated with a random variable with no first moment.

### Definition and first properties

In what follows we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All random variables are defined on this probability space.

For any real-valued random variable $X$, we denote by $\phi_X$ the characteristic function of $X$ defined by:

$$\forall t \in \mathbb{R}, \phi_X(t) = \mathbb{E}[\exp(itX)].$$

1. Prove that $\phi_X$ is a well-defined complex-valued function bounded by 1.

2. Prove that $\forall t, h \in \mathbb{R}, |\phi_X(t+h) - \phi_X(t)| \leq \mathbb{E}[|\exp(ihX) - 1|]$. Deduce that $\phi_X$ is uniformly continuous.

3. Let $p \in \mathbb{N}^*$. Prove that if $X$ is in $L^p(\Omega, \mathcal{F}, \mathbb{P})$ then $\phi_X$ is a function of class $C^p$ with

$$\phi_X(t) = \sum_{k=0}^{p} \frac{i^k}{k!} \mathbb{E}[X^k] t^k + o(t^p).$$

### A partial reciprocal

Let us consider a real-valued random variable $X$ whose characteristic function $\phi_X$ is twice differentiable in a neighborhood of 0.

4. Prove that

$$\forall t \in \mathbb{R}, \frac{2 - \phi_X(t) - \phi_X(-t)}{t^2} = 2\mathbb{E}\left[\frac{1 - \cos(tX)}{t^2}\right].$$

5. Deduce that $\mathbb{E}[X^2] \leq -\phi_X''(0)$.
   *Hint: Use Fatou's lemma.*

6. Conclude that $X$ is in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and that $\phi_X$ is in fact a function of class $C^2$ with $\phi_X''(0) = -\mathbb{E}[X^2]$.

Let us now consider $k \in \mathbb{N}^*$.

Let us consider a real-valued random variable $X$ in $L^{2k-2}(\Omega, \mathcal{F}, \mathbb{P})$ whose characteristic function $\phi_X$ is $2k$ times differentiable in a neighborhood of 0. We also assume $\mathbb{P}(X \neq 0) > 0$.

7. Show that there exists a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ with Radon-Nykodym derivative given by:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{X^{2k-2}}{\mathbb{E}[X^{2k-2}]}.$$

8. Prove that $\phi : t \in \mathbb{R} \mapsto \mathbb{E}^{\mathbb{Q}}[\exp(itX)]$ is twice differentiable in a neighborhood of 0.

9. Conclude that $X$ is in $L^{2k}(\Omega, \mathcal{F}, \mathbb{P})$.

Let us consider a real-valued random variable $X$ whose characteristic function $\phi_X$ is $2k$ times differentiable in a neighborhood of 0.

10. Prove that $X$ is in $L^{2k}(\Omega, \mathcal{F}, \mathbb{P})$ and that $\phi_X$ is in fact a function of class $C^{2k}$.

## An interlude on trigonometric series

Let $t \in (0, \pi)$ and $n \in \mathbb{N}$.

Let us define $S_n(t) = \sum_{k=0}^{n} \sin(kt)$.

11. Prove that

$$S_n(t) = \frac{\sin\left(\frac{nt}{2}\right)\sin\left(\frac{(n+1)t}{2}\right)}{\sin\left(\frac{t}{2}\right)}.$$

12. Deduce that $S_n(t) \leq \frac{1}{\sin\left(\frac{t}{2}\right)} \leq \frac{\pi}{t}$.

Let us consider a positive and non increasing sequence $(b_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} b_n = 0$.

13. Prove that for all $p, q \in \mathbb{N}^*$ with $p < q$, we have the following Abel's summation formula:

$$\sum_{n=p}^{q} b_n \sin(nt) = b_{q+1}S_q(t) - b_p S_{p-1}(t) + \sum_{n=p}^{q}(b_n - b_{n+1})S_n(t).$$

14. Deduce that the series $\sum_{n\geq 0} b_n \sin(nt)$ is convergent and that for all $p \in \mathbb{N}^*$

$$\sum_{n=p}^{+\infty} b_n \sin(nt) = -b_p S_{p-1}(t) + \sum_{n=p}^{+\infty}(b_n - b_{n+1})S_n(t).$$

Let us also define for $n \in \mathbb{N}$, $\beta_n = \sup_{k\geq n} kb_k$.

Let $N_t = \lfloor \frac{\pi}{t} \rfloor$.

15. Prove that for all $m \in \mathbb{N}^*$

$$\left|\sum_{n=m}^{+\infty} b_n \sin(nt)\right| \leq \pi\beta_m + \left|\sum_{n=m+N_t}^{+\infty} b_n \sin(nt)\right|.$$

16. Deduce that for all $m \in \mathbb{N}^*$,

$$\left|\sum_{n=m}^{+\infty} b_n \sin(nt)\right| \leq (\pi + 1)\beta_m + \frac{\pi}{t} b_{m+N_t}.$$

*Hint: use again Abel's summation formula.*

17. Conclude that for all $m \in \mathbb{N}^*$,

$$\left|\sum_{n=m}^{+\infty} b_n \sin(nt)\right| \leq (\pi + 2)\beta_m.$$

Assume now that the sequence $(b_n)_{n\in\mathbb{N}}$ satisfy $\lim_{n\to+\infty} nb_n = 0$.

18. Prove that the series of function
$$\sum_{n\geq 0}^{+\infty} b_n \sin(nt)$$
converges uniformly on $\mathbb{R}$.

**A counterexample to the link between moments and regularity**

Let us consider a $\mathbb{Z}$-valued random variable $X$ with the following distribution:
$$\mathbb{P}(|X| \leq 1) = 0 \text{ and } \forall k \in \mathbb{N} \setminus \{0,1\}, \mathbb{P}(X = -k) = \mathbb{P}(X = k) = \frac{c}{k^2 \log(k)},$$
where $c^{-1} = 2\sum_{k=2}^{+\infty} \frac{1}{k^2\log(k)}$.

19. Prove that $\mathbb{E}[|X|] = +\infty$.

20. Prove that $\forall t \in \mathbb{R}, \phi_X(t) = 2c\sum_{k=2}^{+\infty} \frac{\cos(kt)}{k^2\log(k)}$.

21. Prove that $\phi_X$ is a function of class $C^1$.

22. Comment.