The characteristic function: injectivity, Lévy's continuity theorem, and the CLT

The goal of this problem is to prove two important results regarding the characteristic function. The first one is an injectivity result: the characteristic function of a random variable characterizes its distribution. The second result is known as Lévy's continuity theorem and relates the convergence in distribution of a sequence of random variables to the convergence of the associated sequence of characteristic functions towards a function continuous at point 0.

These results are part of the basic toolbox of anybody interested in probability. In particular, they are essential to prove the notorious Central Limit Theorem (CLT).

In this problem we concentrate on real-valued random variables for the sake of simplicity but it is noteworthy that the results generalize easily to $\mathbb{R}^d$-valued random variables.

Definition and first properties

For any real-valued random variable X, we denote by ϕ_X the characteristic function of X defined by:

$$\forall t \in \mathbb{R}, \phi_X(t) = \mathbb{E}[\exp(itX)].$$

1. Prove that ϕ_X is a well-defined complex-valued function bounded by 1.

2. Prove that $\forall t, h \in \mathbb{R}, |\phi_X(t+h) - \phi_X(t)| \leq \mathbb{E}[|\exp(ihX) - 1|]$. Deduce that ϕ_X is uniformly continuous.

3. Let $p \in \mathbb{N}^*$. Prove that if $X \in L^p$ then ϕ_X is a function of class C^p with

$$\phi_X(t) = \sum_{k=0}^{p} \frac{i^k}{k!} \mathbb{E}[X^k] t^k + o(t^p).$$

The Gaussian case

Let X be a Gaussian $\mathcal{N}(0,1)$ random variable.

4. Prove that ϕ_X is a function of class C^1 and that $\forall t \in \mathbb{R}, \phi'_X(t) = -t\phi_X(t)$.

5. Deduce that $\forall t \in \mathbb{R}, \phi_X(t) = \exp\left(-\frac{t^2}{2}\right)$.

Let X be a Gaussian $\mathcal{N}(\mu, \sigma^2)$ random variable.

6. Prove that $\forall t \in \mathbb{R}, \phi_X(t) = \exp\left(it\mu - \frac{1}{2}\sigma^2 t^2\right)$.

Injectivity

Let Y be a real-valued random variable distributed according to a probability measure μ. Let X be a $\mathcal{N}(0,1)$ random variable independent of Y.

For $n \in \mathbb{N}^*$, we define $Z_n = Y + \frac{X}{n}$.

Let us consider a continuous and bounded function $\psi : \mathbb{R} \to \mathbb{R}$.

7. Prove that
$$\mathbb{E}[\psi(Z_n)] = \frac{n}{\sqrt{2\pi}} \int_{z\in\mathbb{R}} \psi(z) \left(\int_{y\in\mathbb{R}} \exp\left(-\frac{1}{2}n^2(z-y)^2\right) \mu(dy) \right) dz.$$

8. Prove that
$$\forall z \in \mathbb{R}, \int_{y\in\mathbb{R}} \exp\left(-\frac{1}{2}n^2(z-y)^2\right) \mu(dy) = \int_{x\in\mathbb{R}} \exp(-izx) \frac{1}{n\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{x^2}{n^2}\right) \phi_Y(x) dx.$$
Hint: Recall that $\exp\left(-\frac{1}{2}n^2(z-y)^2\right) = \phi_{nX}(z-y) = \phi_{nX}(y-z)$.

9. Prove that $\lim_{n\to+\infty} \mathbb{E}[\psi(Z_n)] = \mathbb{E}[\psi(Y)]$.

10. Deduce that
$$\int_{z\in\mathbb{R}} \psi(z)\mu(dz) = \lim_{n\to+\infty} \frac{1}{2\pi} \int_{z\in\mathbb{R}} \psi(z) \left(\int_{x\in\mathbb{R}} \exp(-izx) \exp\left(-\frac{1}{2}\frac{x^2}{n^2}\right) \phi_Y(x) dx \right) dz.$$

Let Y_1 and Y_2 be two real-valued random variables with the same characteristic function.

11. Prove that Y_1 and Y_2 have the same distribution.

Lévy's continuity theorem

In what follows we will need a classical compactness result for probability measures on $\mathbb{R}$ (implicity equipped with the Borelians) which is a special case of Prokhorov's Theorem:

Theorem 1. *Let us consider a sequence $(\nu_n)_{n\in\mathbb{N}}$ of probability measures on $\mathbb{R}$. Assume that the sequence is tight in the sense that*
$$\forall \epsilon > 0, \exists A > 0, \forall n \in \mathbb{N}, \nu_n(\mathbb{R} \setminus [-A, A]) \leq \epsilon.$$
Then, there exists a probability measure ν on $\mathbb{R}$ and a subsequence $(\nu_{n_k})_{k\in\mathbb{N}}$ of $(\nu_n)_{n\in\mathbb{N}}$ such that $(\nu_{n_k})_{k\in\mathbb{N}}$ weakly converges towards ν, i.e. for all continuous and bounded functions $\psi : \mathbb{R} \to \mathbb{R}$,
$$\lim_{k\to+\infty} \int_{x\in\mathbb{R}} \psi(x)\nu_{n_k}(dx) = \int_{x\in\mathbb{R}} \psi(x)\nu(dx).$$
This result of functional analysis and measure theory is admitted here.

Let us consider a sequence $(X_n)_{n\in\mathbb{N}}$ of real-valued random variables. For all $n \in \mathbb{N}$ let us denote by μ_n and ϕ_n respectively the probability measure on $\mathbb{R}$ associated with X_n and the characteristic function of X_n.

We assume that $(\phi_n)_{n\in\mathbb{N}}$ converges pointwise towards a function ϕ. We also assume that ϕ is continuous at 0.

12. Prove that
$$\forall a > 0, \frac{1}{2a} \int_{-a}^{a} \phi_n(t)dt = \int_{x\in\mathbb{R}} \frac{\sin(ax)}{ax} \mu_n(dx).$$

13. Prove that $\forall \epsilon > 0, \exists \eta > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0$,
$$\int_{x\in\mathbb{R}} \left(1 - \frac{\sin(\eta x)}{\eta x}\right) \mu_n(dx) = \frac{1}{2\eta} \int_{-\eta}^{\eta} (1 - \phi_n(t))dt \leq \frac{\epsilon}{2}.$$

14. Deduce that
$$\forall \epsilon > 0, \exists \eta > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \mu_n(\mathbb{R} \setminus [-2/\eta, 2/\eta]) \leq \epsilon.$$

15. Prove that
$$\forall \epsilon > 0, \exists A > 0, \forall n \in \mathbb{N}, \mu_n(\mathbb{R} \setminus [-A, A]) \leq \epsilon.$$

16. Prove that there exists a probability measure μ on $\mathbb{R}$ and a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ of $(\mu_n)_{n\in\mathbb{N}}$ such that $(\mu_{n_k})_{k\in\mathbb{N}}$ weakly converges towards μ and that ϕ is the characteristic function associated with any random variable with distribution μ.

17. Prove that in fact $(\mu_n)_{n\in\mathbb{N}}$ weakly converges towards the unique probability measure μ such that ϕ is the characteristic function associated with any random variable with distribution μ.

18. Conclude that $(X_n)_{n\in\mathbb{N}}$ converges in distribution towards any random variable with distribution μ.

This beautiful result is called Lévy's continuity theorem.

Central Limit Theorem

Let us consider a sequence $(X_n)_{n\in\mathbb{N}}$ of real-valued random variables. Assume they are i.i.d and in L^2.

Let us denote by μ and σ^2 their common expected value and variance.

For all $n \in \mathbb{N}^*$, let us introduce $Z_n = \sqrt{n}\left(\frac{X_1+\cdots+X_n}{n} - \mu\right)$.

19. Prove that for all $n \in \mathbb{N}^*$, the characteristic function of Z_n is
$$\phi_n : t \mapsto \phi_{X_1-\mu}\left(\frac{t}{\sqrt{n}}\right)^n.$$

20. Prove that $(\phi_n)_{n\in\mathbb{N}^*}$ converges pointwise towards the function $\phi : t \mapsto \exp\left(-\frac{1}{2}\sigma^2 t^2\right)$.

21. Conclude.